# TEMPERATURE FIELD IN INFINITE HOLLOW CYLINDERS HEATED ASYMMETRICALLY AROUND THEIR PERIMETER UNDER GENERAL NONUNIFORM BOUNDARY CONDITIONS

M. K. Kleiner

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A solution has been found, by the method of finite integral transformations, of the heat conduction equation for a hollow cylinder, heated asymmetrically around its perimeter, under general boundary conditions. Formulas are given which reduce the problem with nonuniform boundary conditions to an equivalent problem with uniform boundary conditions.

Solutions have been given in [1, 2] of the heat conduction for hollow cylinders heated symmetrically around their perimeter, under various nonuniform boundary conditions. These solutions were obtained by the method of finite integral transformations, and therefore, as has been shown by Greenberg [3], they cannot satisfy nonuniform boundary conditions at the boundary. Substitutions were given in [4] which allow a problem with nonuniform boundary conditions to be reduced in certain cases to a problem with uniform boundary conditions. We note, however, that of the six expressions given, only two (1 and 3*) satisfy the nonuniform boundary conditions on both surfaces. There are evidently errors in the remaining expressions.

In the industrial practice, especially in the tube manufacturing industry, massive tubes are often heated asymmetrically around their perimeter.

There is therefore interest in solving the following problem of heat conduction in hollow cylinders under general nonuniform boundary conditions:

$$\frac{1}{a}\frac{\partial t(r,\varphi,\tau)}{\partial\tau}=\frac{\partial^2 t(r,\varphi,\tau)}{\partial r^2}+\frac{1}{r}\frac{\partial t(r,\varphi,\tau)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 t(r,\varphi,\tau)}{\partial\varphi^2}+L(\tau)t(r,\varphi,\tau)+\frac{Q(r,\varphi,\tau)}{\lambda},\quad(1)$$

$$a_1\frac{\partial t(R_1,\varphi,\tau)}{\partial r}-b_1 t(R_1,\varphi,\tau)=k_1\psi_1(\varphi,\tau),\quad(2)$$

$$a_2\frac{\partial t(R_2,\varphi,\tau)}{\partial r}+b_2 t(R_2,\varphi,\tau)=k_2\psi_2(\varphi,\tau),\quad(3)$$

$$t(r,0,\tau)=t(r,2\pi,\tau),\quad(4)$$

$$t(r,\varphi,0)=t_0(r,\varphi),\quad(5)$$
$$R_1\leqslant r\leqslant R_2,\quad \tau>0.$$

We shall solve the problem by the method of finite integral transformations [5]. For the solution obtained to satisfy the nonuniform boundary conditions, we shall represent the desired function in the form

$$t(r,\varphi,\tau)=u(r,\varphi,\tau)+z(r)\psi_1(\varphi,\tau)+f(r)[\psi_2(\varphi,\tau)-k_3\psi_1(\varphi,\tau)],\quad(6)$$

where $u(r,\varphi,\tau)$ must satisfy the uniform boundary conditions, while the functions $z(r)$ and $f(r)$ must be continuous in the interval $[R_1, R_2]$, and defined in such a way that $t(r,\varphi,\tau)$ satisfies the nonuniform boundary conditions.

Substituting (6) into (1)–(5), we obtain a system for determining the function $u(r,\varphi,\tau)$.

$$\frac{1}{a}\frac{\partial u(r,\varphi,\tau)}{\partial\tau}=\frac{\partial^2 u(r,\varphi,\tau)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,\varphi,\tau)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u(r,\varphi,\tau)}{\partial\varphi^2}+L(\tau)u(r,\varphi,\tau)+F(r,\varphi,\tau),\quad(7)$$

$$a_1\frac{\partial u(R_1,\varphi,\tau)}{\partial r}-b_1 u(R_1,\varphi,\tau)=0,\quad(8)$$

$$a_2\frac{\partial u(R_2,\varphi,\tau)}{\partial r}+b_2 u(R_2,\varphi,\tau)=0,\quad(9)$$

$$u(r,0,\tau)=u(r,2\pi,\tau),\quad(10)$$

$$u(r,\varphi,0)=t_0(r,\varphi)-z(r)\psi_1(\varphi,0)-f(r)[\psi_2(\varphi,0)-k_3\psi_1(\varphi,0)]=u_0(r,\varphi),\quad(11)$$

where

$$F(r,\varphi,\tau)=\frac{Q(r,\varphi,\tau)}{\lambda}-\frac{1}{a}\left[z(r)\frac{\partial\psi_1}{\partial\tau}+f(r)\times\left(\frac{\partial\psi_2}{\partial\tau}-k_3\frac{\partial\psi_1}{\partial\tau}\right)\right]+z(r)\left[L(\tau)\psi_1+\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\varphi^2}\right]+f(r)\left[(\psi_2-k_3\psi_1)L(\tau)+\frac{1}{r^2}\left(\frac{\partial^2\psi_2}{\partial\varphi^2}-k_3\frac{\partial^2\psi_1}{\partial\varphi^2}\right)\right]+(\psi_2-k_3\psi_1)\left(\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}\right)+\psi_1\left(\frac{d^2z}{dr^2}+\frac{1}{r}\frac{dz}{dr}\right).\quad(12)$$

Then the functions $z(r)$ and $f(r)$ must satisfy the following boundary conditions:

$$r=R_1,\quad a_1\frac{dz}{dr}\psi_1+a_1\frac{df}{dr}\times(\psi_2-k_3\psi_1)-b_1 z\psi_1-b_1 f(\psi_2-k_3\psi_1)=k_1\psi_1,\quad(13)$$

$$r=R_2,\quad a_2\frac{dz}{dr}\psi_1+a_2\frac{df}{dr}\times(\psi_2-k_3\psi_1)+b_2 z\psi_1+b_2 f(\psi_2-k_3\psi_1)=k_2\psi_2.\quad(14)$$

*This is the numbering of [4].

We note that:
with boundary conditions of the first kind

$$-b_1 = b_2 = k_1 = k_2 = 1, \quad a_1 = a_2 = 0,$$

$\psi_i$ (i = 1, 2) are the temperatures of the surfaces;
with boundary conditions of the second kind

$$a_1 = a_2 = \lambda, \quad b_1 = b_2 = 0, \quad -k_1 = k_2 = 1,$$

$\psi_i$ are the surface heat fluxes;
with boundary conditions of the third kind

$$a_1 = a_2 = \lambda, \quad b_1 = \alpha_1, \quad b_2 = k_2 = \alpha_2, \quad k_1 = -\alpha_1,$$

$\psi_i$ are the temperatures of the surrounding media.

Following [6], we shall try to make the chosen functions z(r) and $f$(r) solutions of the expressions enclosed in the last two parentheses of (12).

We give expressions for the functions z(r) and $f$(r) below.

1. The boundary conditions are of the first kind ($\psi_2 \neq \psi_1$) on both surfaces,

$$z(r) = 1, \quad f(r) = \frac{\ln(R_1/r)}{\ln\omega}, \quad k_3 = 1.$$

2. The boundary conditions are of the second kind ($\psi_2 \neq \psi_1$) on both surfaces,

$$z(r) = \frac{r^2}{2\lambda R_2}\frac{1}{1-\omega} - \frac{\omega R_2}{\lambda}\frac{\ln r}{1-\omega},$$

$$f(r) = \frac{r^2}{2\lambda R_2}\frac{1}{1-\omega^2} - \frac{\omega^2 R_2}{\lambda}\frac{\ln r}{1-\omega^2}, \quad k_3 = 1.$$

3. The boundary conditions are of the third kind ($\psi_2 \neq \psi_1$) on both surfaces,

$$z(r) = 1, \; f(r) = \left[\frac{\lambda}{\alpha_1\delta}\frac{1-\omega}{\omega} - \ln(R_1/r)\right] \Big/ \left[\frac{\lambda}{\alpha_1\delta} \times \frac{1-\omega}{\omega} + \frac{\lambda}{\alpha_2\delta}(1-\omega) - \ln\omega\right], \quad k_3 = 1.$$

4. Mixed boundary conditions of the first and second kinds:

a) on the inner surface—first kind, on the outer—second kind,

$$z(r) = 1, \quad f(r) = -(R_2/\lambda)\ln(R_1/r), \quad k_3 = 0,$$

b) on the inner surface—second kind, on the outer—first kind,

$$z(r) = -\frac{\omega R_2}{\lambda}\ln\frac{\omega r}{R_1}, \quad f(r) = 1, \quad k_3 = 0.$$

5. Mixed boundary conditions of the first and third kind ($\psi_2 \neq \psi_1$):

a) on the inner surface—first kind, on the outer—third kind,

$$z(r) = 1,$$

$$f(r) = \ln(R_1/r) \Big/ \left[\ln\omega - \frac{\lambda}{\alpha_2\delta}(1-\omega)\right], \quad k_3 = 1,$$

b) on the inner surface—third kind, on the outer—first kind,

$$z(r) = 1, \quad f(r) = \left[\ln R_1/r - \frac{\lambda}{\alpha_1\delta}\frac{1-\omega}{\omega}\right] \Big/ \left[\ln\omega - \frac{\lambda}{\alpha_1\delta}\frac{1-\omega}{\omega}\right], \quad k_3 = 1.$$

6. Mixed boundary conditions of the second the third kinds:

a) on the inner surface—second kind, on the outer—third kind,

$$z(r) = -\frac{\omega R_2}{\lambda}\ln\omega r/R_1 + \frac{\omega}{\alpha_2}, \quad f(r) = 1, \quad k_3 = 0;$$

b) on the inner surface—third kind, on the outer—second kind,

$$z(r) = 1, \quad f(r) = \frac{R_2}{\lambda}\ln r/R_1 + \frac{1}{\alpha_1\omega}, \quad k_3 = 0.$$

The expressions for z(r) and $f$(r) presented in paragraphs 1 to 6 also cover the special cases when uniform boundary conditions are assigned on one of the surfaces.

If $\psi_1 = \psi_2 \equiv 0$, we may of course put $z(r) = f(r) \equiv 0$.

If $\psi_1 = \psi_2 = \psi \neq 0$, $\psi_1$ and $\psi_2$ having the same dimension (cases 1, 2, 3, 5a and 5b), we must put $f(r) = 0$. In all the remaining cases the values of z(r), $f$(r), and $k_3$ stay unchanged.

All the values of z(r) and $f$(r), besides those given in paragraph 2, are solutions of the differential expressions enclosed in the last two curved brackets of (12). The values of these expressions for the case of paragraph 2 are constant.

Eliminating the differential operators with respect to $\varphi$ with the aid of an integral transformation in the interval $[0, 2\pi]$ with the kernel [5]

$$K(\nu, \varphi) = \begin{cases} \frac{1}{\pi\varepsilon_m}\cos m\varphi \text{ when } \nu = 2m \\ \frac{1}{\pi}\sin m\varphi \text{ when } \nu = 2m+1 \end{cases}, \quad (15)$$

$$\varepsilon_m = 2 \text{ when } m = 0 \text{ and } \varepsilon_m = 1 \text{ when } m \neq 0,$$

m = 0, 1, 2, 3, ..., we shall reduce our problem (7)–(11) to the form

$$\frac{\partial^2 u_\nu(r,\tau)}{\partial r^2} + \frac{1}{r}\frac{\partial u_\nu(r,\tau)}{\partial r} - \frac{m^2}{r^2}u_\nu(r,\tau) = \frac{1}{a}\frac{\partial u_\nu(r,\tau)}{\partial\tau} - L(\tau)u_\nu(r,\tau) - F_\nu(r,\tau), \quad (16)$$

$$a_1\frac{\partial u_\nu(R_1,\tau)}{\partial r} - b_1 u_\nu(R_1,\tau) = 0, \quad (17)$$

$$a_2\frac{\partial u_\nu(R_2,\tau)}{\partial r} + b_2 u_\nu(R_2,\tau) = 0, \quad (18)$$

$$u_\nu(r, 0) = u_{\nu 0}(r), \quad (19)$$

where

$$u_\nu(r,\ \tau)=\int\limits_0^{2\pi} u(r,\ \varphi,\ \tau)K(\nu,\ \varphi)\,d\varphi,$$

$$F_\nu(r,\ \tau)=\int\limits_0^{2\pi} F(r,\ \varphi,\ \tau)K(\nu,\ \varphi)\,d\varphi,$$

$$u_{\nu 0}(r)=\int\limits_0^{2\pi} u_0(r,\ \varphi)K(\nu,\ \varphi)\,d\varphi.$$

The kernel of the integral transformation which permits us to eliminate differential operations with respect to r in (16)–(18) will be

$$M_m(n,\ r)=\frac{1}{C_{m,n}}rM_{m,n}(r), \tag{20}$$

where $M_{m,n}(r)$ is an eigenfunction of the problem

$$\frac{d^2M_{m,n}(r)}{dr^2}+\frac{1}{r}\frac{dM_{m,n}(r)}{dr}+\left(\beta_{m,n}^2-\frac{m^2}{r^2}\right)M_{m,n}(r)=0, \tag{21}$$

$$a_1\frac{dM_{m,n}(R_1)}{dr}-b_1M_{m,n}(R_1)=0, \tag{22}$$

$$a_2\frac{dM_{m,n}(R_2)}{dr}+b_2M_{m,n}(R_2)=0. \tag{23}$$

The solution of (21) gives the cylindrical functions

$$M_{m,n}(r)=A_{m,n}I_m\left(\mu_{m,n}\frac{r}{R_2}\right)+B_{m,n}N_m\left(\mu_{m,n}\frac{r}{R_2}\right), \tag{24}$$

where the $I_m(x)$ are Bessel functions of the first kind and order m; $N_m(x)$ is a Neumann function of order m, and $\mu_{m,n}=\beta_{m,n}R_2$.

Substituting (24) into (22) and (23), we obtain the characteristic equation for the eigenvalues $\mu_{m,n}$

$$\left[\left(\frac{a_1m}{\omega}-\frac{b_1\delta}{1-\omega}\right)I_m(\mu_{m,n}\omega)-a_1\mu_{m,n}I_{m+1}(\mu_{m,n}\omega)\right]\times$$
$$\times\left[\left(a_2m+\frac{b_2\delta}{1-\omega}\right)N_m(\mu_{m,n})-a_2\mu_{m,n}N_{m+1}(\mu_{m,n})\right]-$$
$$-\left[\left(\frac{a_1m}{\omega}-\frac{b_1\delta}{1-\omega}\right)N_m(\mu_{m,n}\omega)-a_1\mu_{m,n}N_{m+1}(\mu_{m,n}\omega)\right]\times$$
$$\times\left[\left(a_2m+\frac{b_2\delta}{1-\omega}\right)I_m(\mu_{m,n})-a_2\mu_{m,n}I_{m+1}(\mu_{m,n})\right]=0. \tag{25}$$

We shall find, similarly, that, to an accuracy up to an arbitrary multiplier, which we may conveniently take equal to unity, the constants $A_{m,n}$ and $B_{m,n}$ in (24) have the form:

$$A_{m,n}=\left(a_2m+\frac{b_2\delta}{1-\omega}\right)N_m(\mu_{m,n})-a_2\mu_{m,n}N_{m+1}(\mu_{m,n}),$$

$$B_{m,n}=-\left[\left(a_2m+\frac{b_2\delta}{1-\omega}\right)I_m(\mu_{m,n})-a_2\mu_{m,n}I_{m+1}(\mu_{m,n})\right] \tag{26}$$

We shall determine the normalizing denominator [5] from the expression

$$C_{m,n}=\int\limits_{R_1}^{R_2} rM_{m,n}^2(r)\,dr=\frac{R_2^2}{2}\Big[Z_m^2(\mu_{m,n})+Z_{m+1}^2(\mu_{m,n})-$$
$$-\frac{2m}{\mu_{m,n}}Z_m(\mu_{m,n})Z_{m+1}(\mu_{m,n})\Big]-\frac{R_1^2}{2}\Big[Z_m^2(\mu_{m,n}\omega)+$$
$$+Z_{m+1}^2(\mu_{m,n}\omega)-\frac{2m}{\mu_{m,n}\omega}Z_m(\mu_{m,n}\omega)Z_{m+1}(\mu_{m,n}\omega)\Big], \tag{27}$$

where

$$Z_p(x)=A_{m,n}I_p(x)+B_{m,n}N_p(x).$$

Since

$$I_{p+1}(x)N_p(x)-I_p(x)N_{p+1}(x)=\frac{2}{\pi x},$$

the expression for $C_{m,n}$ is appreciably simplified and takes the form

$$C_{m,n}=\frac{2}{\pi^2}\Big\{a_2^2+\frac{1}{\mu_{m,n}^2}\left(a_2m+\frac{b_2\delta}{1-\omega}\right)^2-$$
$$-\frac{2ma_2}{\mu_{m,n}^2}\left(a_2m+\frac{b_2\delta}{1-\omega}\right)-P_{m,n}\Big[a_1^2+\frac{1}{\mu_{m,n}^2}\left(\frac{ma_1}{\omega}-\frac{b_1\delta}{1-\omega}\right)^2-$$
$$-\frac{2ma_1}{\mu_{m,n}^2}\left(\frac{a_1m}{\omega}-\frac{b_1\delta}{1-\omega}\right)\Big]\Big\}, \tag{28}$$

where

$$P_{m,n}=\left[\left(a_2m+\frac{b_2\delta}{1-\omega}\right)I_m(\mu_{m,n})-a_2\mu_{m,n}I_{m+1}(\mu_{m,n})\right]^2\Big/$$
$$\Big/\left[\left(\frac{a_1m}{\omega}-\frac{b_1\delta}{1-\omega}\right)I_m(\mu_{m,n}\omega)-a_1\mu_{m,n}I_{m+1}(\mu_{m,n}\omega)\right]^2.$$

Having accomplished the integral transformation of the problem (16)–(18) in the interval $[R_1, R_2]$ with kernel (20), we obtain an ordinary differential equation of the first order

$$\frac{du_{\nu,n}(\tau)}{d\tau}+a\left[\frac{\mu_{m,n}^2}{R_2^2}-L(\tau)\right]u_{\nu,n}(\tau)=aF_{\nu,n}(\tau), \tag{29}$$

with the initial condition

$$u_{\nu,n}(0)=\int\limits_{R_1}^{R_2} u_{\nu 0}(r)M_m(n,\ r)\,dr. \tag{30}$$

The solution of (29), allowing for (30) will be

$$u_{\nu,n}(\tau)=\exp\left[-\mu_{m,n}^2\frac{a\tau}{R_2^2}+a\int\limits_0^\tau L(\tau)\,d\tau\right]\Big\{a\int\limits_0^\tau F_{\nu,n}(\tau)\times$$
$$\times\exp\left[\mu_{m,n}^2\frac{a\tau}{R_2^2}-a\int\limits_0^\tau L(\tau)\,d\tau\right]d\tau+u_{\nu,n}(0)\Big\}.$$

Carrying out the inverse transformations, we obtain

$$u_\nu(r,\ \tau)=\sum_{n=1}^{\infty}u_{\nu,n}(\tau)M_{m,n}(r),$$

and finally,

$$u(r, \varphi, \tau) = \sum_{m=0}^{\infty} \cos m\varphi \left[ \sum_{n=1}^{\infty} u_{2m,n}(\tau) M_{m,n}(r) \right] + \sum_{m=1}^{\infty} \sin m\varphi \left[ \sum_{n=1}^{\infty} u_{2m+1,n}(\tau) M_{m,n}(r) \right], \quad (31)$$

where

$$u_{2m,n} = \exp\left[ -\mu_{m,n}^2 \frac{a\tau}{R_2^2} + a\int_0^{\tau} L(\tau)\,d\tau \right] \times \left\{ a \int_0^{2\pi}\int_{R_1}^{R_2}\int_0^{\tau} F(r, \varphi, \tau) \frac{\cos m\varphi}{\pi\varepsilon_m} M_m(n, r) \times \exp\left[ \mu_{m,n}^2 \frac{a\tau}{R_2^2} - a\int_0^{\tau} L(\tau)\,d\tau \right] d\varphi\, dr\, d\tau + \int_0^{2\pi}\int_{R_1}^{R_2} u_0(r, \varphi) \frac{\cos m\varphi}{\pi\varepsilon_m} M_m(n, r)\, d\varphi\, dr \right\}, \quad (32)$$

$$u_{2m+1,n} = \exp\left[ -\mu_{m,n}^2 \frac{a\tau}{R_2^2} + a\int_0^{\tau} L(\tau)\,d\tau \right] \left\{ a \int_0^{2\pi}\int_{R_1}^{R_2}\int_0^{\tau} F(r, \varphi, \tau) \times \frac{\sin m\varphi}{\pi} M_m(n, r) \exp\left[ \mu_{m,n}^2 \frac{a\tau}{R_2^2} - a\int_0^{\tau} L(\tau)\,d\tau \right] d\varphi\, dr\, d\tau + \int_0^{2\pi}\int_{R_1}^{R_2} u_0(r, \varphi) \frac{\sin m\varphi}{\pi} M_m(n, r)\, d\varphi\, dr \right\}. \quad (33)$$

$M_m(n, r)$ is determined from (20), taking account of (24), (26), and (28); $F(r, \varphi, \tau)$—from (12); $u_0(r, \varphi)$—from (11).

Thus, the temperature field of the hollow cylinder in the general case is determined by expression (6), taking account of (31)–(33).

If, besides (4), we are given the condition

$$t(r, \varphi, \tau) = t(r, -\varphi, \tau),$$

which in the general case must be satisfied also by the functions $Q(r, \varphi, \tau)$, $\psi_i(\varphi, \tau)$, and $t_0(r, \varphi)$, then the second sum in (31), containing $\sin m\varphi$, is equal to zero, since the kernel $L(\nu, \varphi)$ with $\nu = 2m + 1$ and $u_{2m+1,n}$ is equal to zero. If the internal heat sources Q and the initial and boundary conditions are independent of angle $\varphi$, then from (31)–(33) we may obtain the corresponding expression for the temperature field of a symmetrically heated hollow cylinder.

In this case, $m \equiv 0$, and then

$$t(r, \tau) = z(r)\psi_1(\tau) + f(r)[\psi_2(\tau) - k_3\psi_1(\tau)] + \sum_{n=1}^{\infty} \exp\left[ -\mu_{0,n}^2 \frac{a\tau}{R_2^2} + a\int_0^{\tau} L(\tau)\,d\tau \right] \left\{ a \int_{R_1}^{R_2}\int_0^{\tau} F(r, \tau) M_0(n, r) \times \exp\left[ \mu_{0,n}^2 \frac{a\tau}{R_2^2} - a\int_0^{\tau} L(\tau)\,d\tau \right] + \int_{R_1}^{R_2} u_0(r) M_0(n, r)\, dr \right\}. \quad (34)$$

Then $\mu_{0,n}$, $M_0(n, r)$ are determined, respectively, from (25) and (20), taking account of (24), (26) and (28) with $m \equiv 0$. The form of the functions $z(r)$ and $f(r)$ remains unchanged.

Expressions (31) and (34) cover all possible combinations of uniform and nonuniform boundary conditions of the first, second, and third kinds, and because the solutions were obtained in the form of (6), they satisfy the nonuniform boundary conditions both inside and on the boundaries of the interval $[R_1, R_2]$.

## NOTATION

$t$, $r$, $\varphi$, $\tau$ are the current temperature, radius, angle, and time; $a$, $\lambda$ are thermal diffusivity and thermal conductivity; $R_1$, $R_2$ are the inner and outer radii of the tube; $\omega = R_1/R_2$; $\alpha_1$, $\alpha_2$ are coefficients of heat transfer from the inner and outer surfaces of the tube; $\delta = R_2 - R_1$ is the tube wall thickness.

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Turbine Institute, Dnepropetrovsk